A Two-Level Trajectory Decomposition Algorithm Featuring Optimal Intermediate Target Selection

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A decomposition algorithm is presented which optimizes complex missions by partitioning the trajectory into natural segments such as ascent or entry. Each segment defines a full-rank targeting subproblem. These are solved sequentially using the Newton-Raphson algorithm. The master problem, representing the complete mission, is to determine subproblem targets and master-problem controls that optimize the mission objective subject to intersegment constraints. The gradient projection algorithm solves this problem using derivatives obtained analytically from finite-difference subproblem sensitivities. Thus, the mission is optimized by coordinating the solution of tractible subproblems. Computational results for a synchronous equatorial mission are included.

Introduction

IN recent years, there has been renewed interest in solution of trajectory problems formulated as discrete parameter nonlinear programs.[1-3] Ascent, orbital, and entry formulations are solved routinely with a variety of standard nonlinear programming algorithms. For example, variable metric techniques using some form of penalty function[4,5] or the accelerated gradient projection method[6] are excellent algorithms for solving most standard trajectory problems and are used in several operational trajectory programs.[7,8] However, computational results with these single-level algorithms on extremely large problems representing multiple flight-regime missions have been disappointing and clearly illustrate the limitations of these standard techniques. This lack of success appears to be related to two important problem characteristics: most missions involve several distinctly different flight regimes, and flight time of most missions is quite long when compared to the duration of any single trajectory segment. The first effect causes convergence problems because the dominant terms change as the vehicle passes through various flight regimes, thereby amplifying the nonlinearity of the total problem. The second characteristic causes scaling problems because of the nearly exponential growth of trajectory sensitivity elements with time. The decomposition approach presented in this paper tends to remove these inherent difficulties by defining subproblems that are contained completely within each flight regime and are significantly shorter in duration.

The decomposition approach is based upon partitioning of the total mission into an ordered sequence of self-contained mission segments such as ascent, re-entry, and so on. The constraints in each mission segment then are used to define an associated full-rank targeting subproblem. Sequential solution of the subproblems insures that the majority of the mission constraints are satisfied on each master-level iteration. A master problem, which represents the complete mission design, then is formulated in terms of the individual subproblem performance functionals, constraints, and control variables. Included in the master problem are intersegment constraints that cannot be satisfied at the subproblem level. (Total ΔV budgets or propellant limits are examples of master-level constraints that couple subproblems.) The master problem is optimized using a two-level procedure, with the master-level algorithm optimally coordinating solution of the subproblems. The dual-level algorithm implemented employs the accelerated gradient-projection algorithm to solve the nonlinear program defined by the master problem, and the Newton-Raphson algorithm to solve the sets of nonlinear equations defined by the subproblems.

A unique consequence of using a two-level algorithm in conjunction with this decomposition approach is that intermediate target values, which are held fixed at the subproblem level, can be used as independent variables to be optimized at the master level. This approach is quite different from other trajectory decomposition strategies,[9,10] which relax state continuity constraints on position and velocity at certain critical points in the trajectory. In these formulations, initial conditions for each segment then become independent variables to be optimized at the master level, and constraints are added to insure state-variable continuity. This technique produces a large number of master-level independent variables for realistically complex missions. An even more serious drawback to this approach is lack of a feasible trajectory satisfying Newton's laws until late in the iterative process. As a result, premature algorithm termination results in an unacceptable trajectory. In contrast, this decomposition approach insures that the dimension of independent variable space in each subproblem is only as large as the minimum number of variables required to meet subproblem constraints; and only subproblem coupling constraints add dimensionality at the master level.

Problem Formulation

Consider a mission consisting of s trajectory segments, as illustrated in Fig. 1. Suppose that each segment has its own physical control vector $\boldsymbol{u}^k$ containing m_k components and its own constraint vector $\boldsymbol{c}^k$ having n_k components. Let $\boldsymbol{v}^k$ be the target value of the constraint vector for segment k. Finally, suppose that there are n constraints C_i, which are best associated with the mission as a whole rather than any particular segment. Let V_i denote the target values of these constraints. The problem then is as follows:

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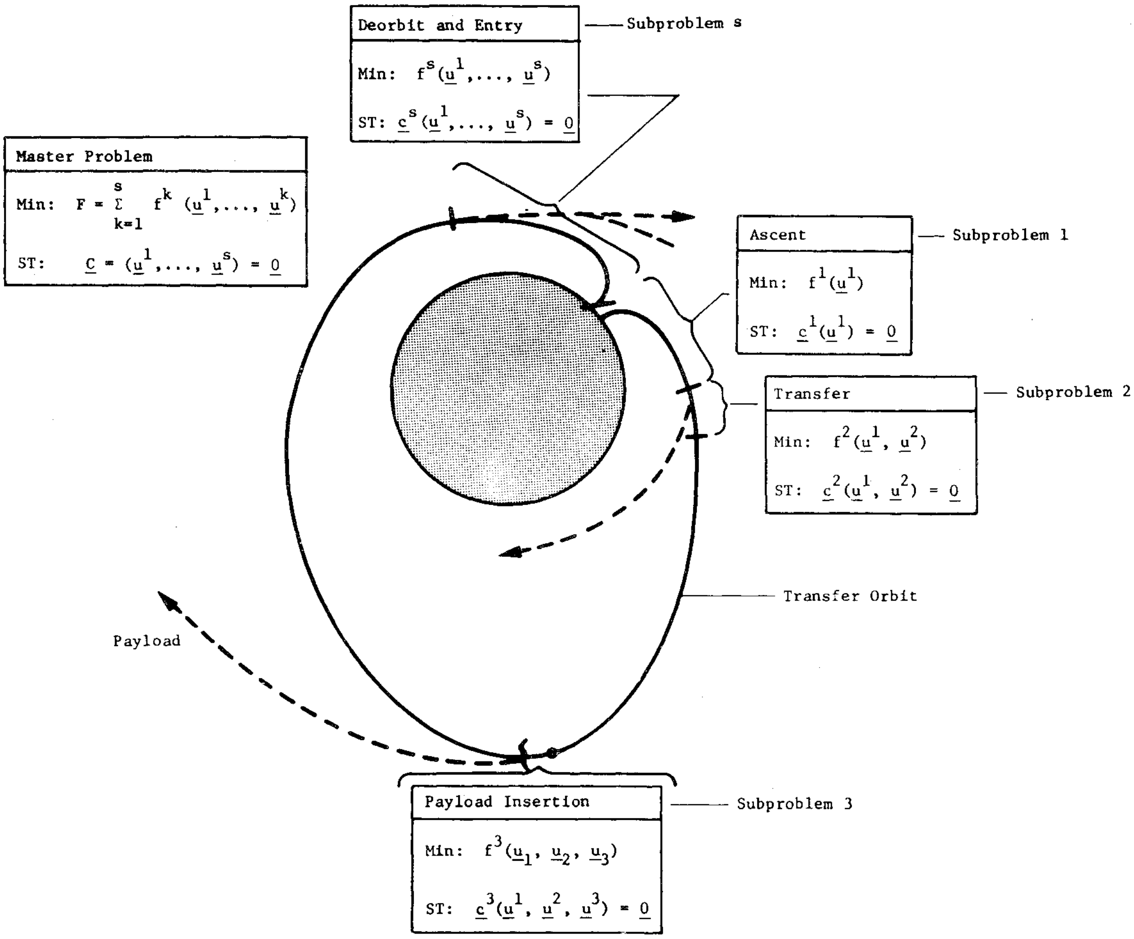

Fig. 1 Basic master-problem/subproblem relationship.

Minimize¶

$$F\Big(\biguplus_{k=1}^{s} u^k \Big) = \sum_{k=1}^{s} f^k \Big(\biguplus_{l=1}^{k} u^l \Big) \tag{1a}$$

subject to

$$v_i^k = c_i^k \Big(\biguplus_{l=1}^{k} u^l \Big) \qquad (\text{for } k=1,\ldots,s;\ \text{for } i=1,\ldots,n_k) \tag{1b}$$

$$V_i = C_i \Big(\biguplus_{k=1}^{s} u^k \Big) \qquad (\text{for } i=1,\ldots,n) \tag{1c}$$

In this formulation, equations of motion are solved directly using numerical integration. This enables state equations for position, velocity, mass, etc., to be eliminated from the problem statement. In addition, the dependent variables f^k, c^k, and C are evaluated at events determined from the iterative solution of the event criteria equations. Inequality constraints can be included in this formulation merely by adding logic to determine active and inactive constraints.

Decomposition Approach

The process of decomposing a problem by partitioning it into full-rank subproblems is based upon two fundamental ploys. The first is the use in each segment of certain key control variables to satisfy the constraints of that segment. Thus, segment k is made into a full-rank subproblem by designating n_k of the physical control vector components as subproblem variables and using them to satisfy the n_k constraints of that segment. The remaining control vector components of segment k are grouped together with similar variables from the other segments. This collection of controls, along with the overall mission objective F and the coupling constraints C, is made into a master problem of minimization subject to equality constraints. This partitioning of the original problem lends itself well to computation. The subproblems, which must be resolved for each new choice of master-problem controls, are solved by the highly efficient Newton-Raphson procedure. The master problem, which serves to coordinate the subproblem solutions, uses the equally efficient but more time-consuming accelerated projected-gradient algorithm.

The second fundamental ploy is use of constraint target values of the various subproblems as master-problem independent variables. To obtain an optimum composite trajectory from a set of mission segments, the mission analyst typically varies the segment aim points parametrically and chooses the endpoint combination that results in the lowest overall cost. Indeed, in trajectory analysis the decision "where to go" is usually more important than "how to get there." By using subproblem constraint targets as master-problem controls, the decomposition procedure automates the analyst's successful design approach.

The successful convergence of the decomposition procedure demands a reasonable partitioning of the original controls and constraints into the master-problem and subproblem categories. To be more precise, the subproblem controls and constraints must be chosen so that each subproblem will have

¶The notation $\biguplus_{k=1}^{s}$ denotes the ordered union of the indexed quantity immediately to its right.

a solution for any set of subproblem constraint target values that reasonably might arise during the master-problem iteration process. Thus, for a given subproblem, the controls that have the most substantial effect on that subproblem's constraint set should be chosen. Similarly, if a particular constraint cannot be assigned to any subproblem whose controls can achieve its satisfaction, it should be designated a master-problem constraint. Finally, the number of master-problem constraints should be held to a minimum. Indeed, the master problem should be kept as simple as possible because each of its iterations requires the re-solution of all of the subproblems.

The decomposition procedure maintains simulation flexibility in obtaining master-problem control sensitivity information by using numerical differencing. Solution of the master problem by descent requires constrained derivatives: quotients of dependent perturbations in the master-problem objectives and constraints by independent perturbations in master-problem controls, assuming that the subproblem controls adjust uniquely to keep the subproblem targeted. These derivatives could be approximated by numerical differencing of master-problem trajectories consisting of iteratively targeted subproblems. This approach, however, must be rejected because it is susceptible to numerical error and is demanding in computational effort. Instead, formulas are used which relate constrained derivatives to partial derivatives of the master-problem objective and constraints with respect to all of the physical controls of the original problem. These partial derivatives are approximated conveniently by numerical differencing of the master-problem trajectories without subproblem targeting. Thus, the need for deriving variational equations for each simulated trajectory is eliminated for a reasonable computational price. Any trajectory that can be simulated can be shaped with no additional analytical effort.

To define the procedure precisely, considerable nomenclature must be established. Most of the user-supplied parameters already have been defined; two, however, remain. The first is p_k, the number of subproblem constraint target values from segment k which are to be used as master-problem independent variables. The second is q_k, the number of master-problem constraints arising from segment k.

Consider next the procedure's working variables. To simplify notation, the subproblem and master-problem controls are given distinct literal symbols and resequenced. Let y_j^k denote the jth subproblem control arising from segment k and z_j^k be the jth master-problem control from that segment. Similarly, the segment constraint target values

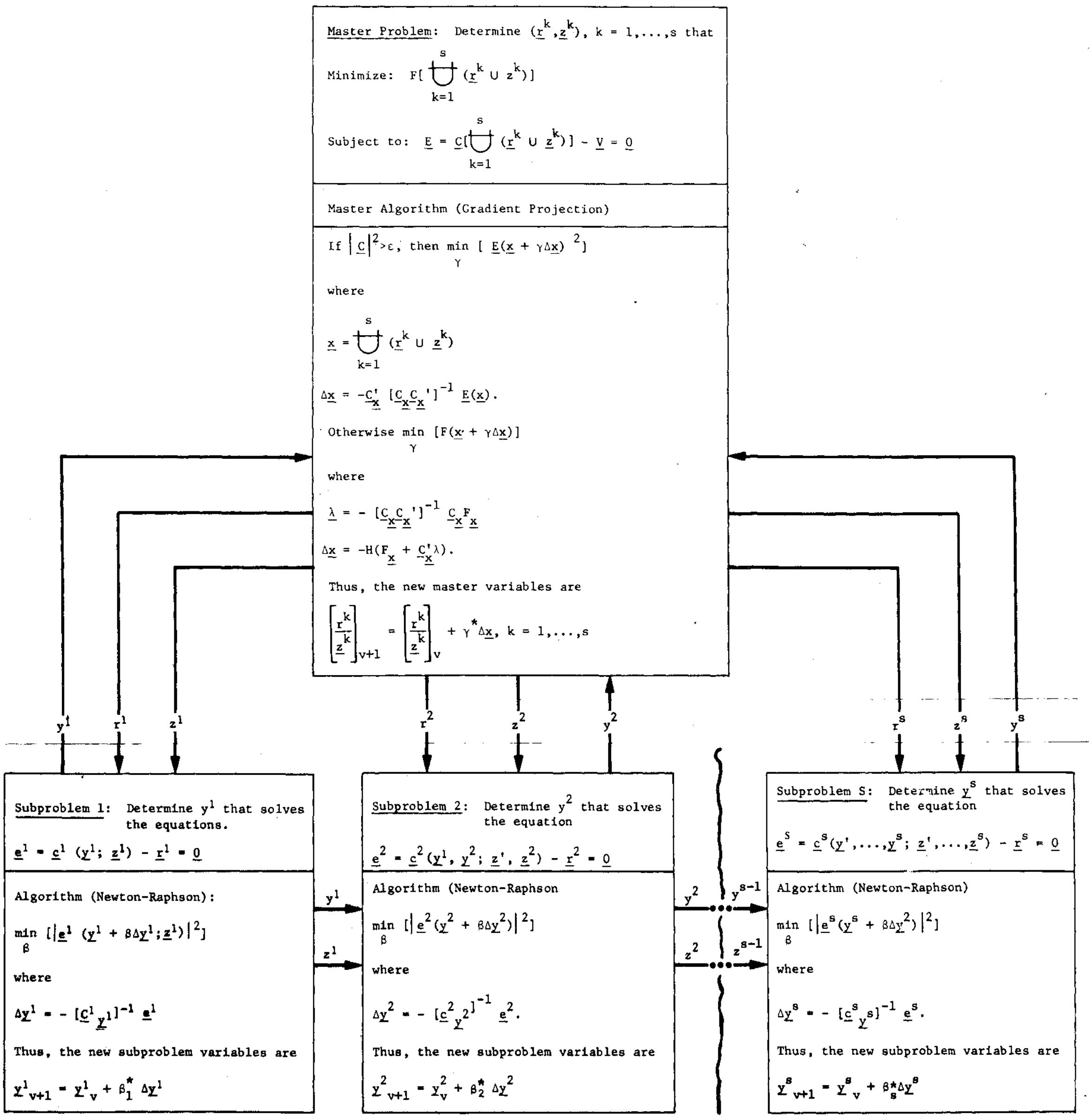

Fig. 2 Illustration of data flow between the master problem and subproblems for this decomposition approach.

are assigned new symbols to distinguish those that are to be held fixed, t_i^k, from those that are to be used as a master-problem independent variables, r_i^k. Finally, the master-problem constraint vector is resequenced so that its first q^l components denoted by C^l arise from the first subproblem, and the next q^2 components denoted by C^2 arise from the second subproblem, and so forth. In terms of this new notation, the original problem (1) becomes as follows:

Minimize

$$F\left[\biguplus_{k=1}^{s}(r^k U z^k)\right]=f\left[\biguplus_{k=1}^{s}(y^k U z^k)\right] \tag{2a}$$

subject to

$$C^l\left[\biguplus_{k=1}^{s}(r^k U z^k)\right]=C^l\left[\biguplus_{k=1}^{s}(y^k U z^k)\right] \quad (\text{for } l=1,\ldots,s) \tag{2b}$$

where r^k is a master-problem independent variable, z^k is a master-problem independent variable, and

$$y^k\left[\biguplus_{k=1}^{s}(r^k U z^k)\right]$$

is the unique vector of subproblem independent variables for subproblem k, satisfying that subproblem's constraint set for the current set of master-problem independent variables.

Two-Level Algorithm

To solve (2), a procedure first must be established for solving the subproblems, that is, for determining the unique y^k given the r^k and z^k. As noted, the Newton-Raphson algorithm for solving full-rank systems of nonlinear equations is the technique selected. To start the iterative solution, the user must input a good estimate $(y^k)_0$ for the physical-control vector of subproblem k, which approximately yields the constraint target values for that subproblem. The procedure, then, successively refines this estimate using the Newton-Raphson recursion formula:

$$(y^k)_{\nu+1}=[y^k-(A^{kk})^{-1}(c^k-v^k)]_\nu \tag{3}$$

where

$$A^{lk}=\frac{\partial c^k}{\partial y^l} \quad (\text{for } l=1,\ldots,s;\ \text{for } k=1,\ldots,l) \tag{4}$$

After the first set of subproblems is solved, the converged subproblem control values from one subproblem set are used as the starting estimates for the next subproblem set. Furthermore, the Jacobian matrix A^{kk} is updated from one iteration to the next only if the old Jacobian does not reduce the constraint errors in norm by a user-specified fraction ρ.

This gradient-projection algorithm developed by Kelley and Speyer[6] is used to minimize the master problem. The interaction between the master and the subproblem algorithms is depicted in Fig. 2.

Gradient Equations

The only new technique involved is the computation of the constrained derivatives in terms of the partial derivatives of the master-problem objective and constraints with respect to all of the physical controls of the original problem. First, the perturbation of the subproblem independent variables caused by perturbations in the master-problem independent variables must be determined. Both master-problem physical controls and subproblem constraint target values must be considered. Once these constrained derivatives of the subproblem independent variables are determined, they can be used to calculate the desired constrained derivatives of the master-problem objective and constraints with respect to all of the master-problem variables.

The constrained derivatives of the subproblem controls are all derived from the basic linearized subproblem equation

$$A^{ll}\delta y^l=\delta r^l-\sum_{k=1}^{l-1}A^{lk}\delta y^k-\sum_{k=1}^{l}B^{lk}\delta z^k \quad (\text{for } l=1,\ldots,s) \tag{5}$$

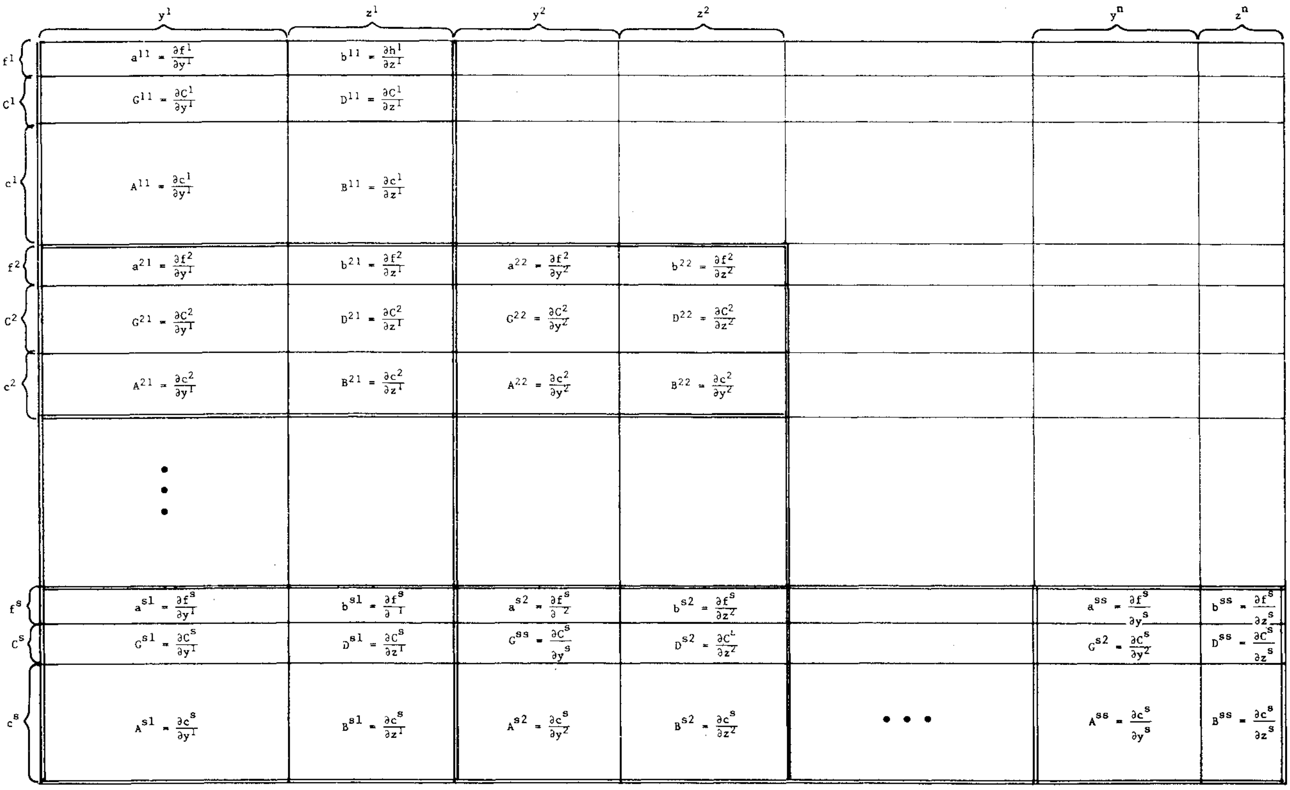

Fig. 3 Composite tableau of objective and constraint sensitivities.

where

$$B^{lk} = \frac{\partial c^l}{\partial z^k} \qquad (\text{for } l=1,\dots,s; \text{ for } k=1,\dots,l) \tag{6}$$

The sensitivity matrix for the nondecomposed problem is given in Fig. 3. This matrix graphically illustrates the structure of the problem and the basis of this decomposition method.

The constrained derivatives with respect to the master-problem physical controls are given by the equation

$$\frac{\delta y^l}{\delta z^k} = -(A^{ll})^{-1}\left[B^{lk} + \sum_{o=k}^{l-1} A^{lo}\frac{\delta y^o}{\delta z^k}\right]$$

$$(\text{for } l=1,\dots,s; \text{ for } k=1,\dots,l) \tag{7}$$

The constrained derivatives with respect to the subproblem constraint target values are computed from the two formulas

$$\frac{\delta y^k}{\delta r^k} = (A^{kk})^{-1}E^{pk}_{n_k} \qquad (\text{for } k=1,\dots,s) \tag{8}$$

where $E^{pk}_{n_k}$ is the matrix consisting of the first p_k columns of the identity matrix of order n_k, and

$$\frac{\delta y^l}{\delta r^k} = -(A^{ll})^{-1}\sum_{o=k}^{l-1} A^{lo}\frac{\delta y^o}{\delta r^k}$$

$$(\text{for } l=1,\dots,s; \text{ for } k=1,\dots,l-1) \tag{9}$$

The constrained derivatives of the master-problem objective with respect to both the master-problem physical controls and the subproblem constraint target values follow from the "chain rule" for differentiation. They are computed as

$$\frac{\delta F}{\delta z^k} = \sum_{l=k}^{s}\left[b^{lk} + \sum_{o=k}^{l} a^{lo}\frac{\delta y^o}{\delta z^k}\right] \qquad (\text{for } k=1,\dots,s) \tag{10}$$

and

$$\frac{\delta F}{\delta r^k} = \sum_{l=k}^{s}\sum_{o=k}^{l} a^{lo}\frac{\delta y^o}{\delta r^k} \qquad (\text{for } k=1,\dots,s) \tag{11}$$

where

$$a^{lk} = \frac{\partial f^l}{\partial y^k} \qquad (\text{for } l=1,\dots,s; \text{ for } k=1,\dots,l) \tag{12}$$

and

$$b^{lk} = \frac{\partial f^l}{\partial z^k} \qquad (\text{for } l=1,\dots,s; \text{ for } k=1,\dots,l) \tag{13}$$

Finally, the constrained derivatives of the master-problem constraints with respect to the master-problem physical controls and the subproblem constraint target values follow from a straightforward application of the "chain rule." They are related to the appropriate partial derivatives by the equations

$$\frac{\delta C^l}{\partial z^k} = D^{lk} + \sum_{o=k}^{l} G^{lo}\frac{\delta y^o}{\delta z^k}$$

$$(\text{for } l=1,\dots,s; \text{ for } k=1,\dots,l) \tag{14}$$

and

$$\frac{\delta C^l}{\delta r^k} = \sum_{o=k}^{l} G^{lo}\frac{\delta y^o}{\delta r^k} \qquad (\text{for } l=1,\dots,s; \text{ for } k=1,\dots,l) \tag{15}$$

where

$$D^{lk} = \frac{\partial C^l}{\partial z^k} \qquad (\text{for } l=1,\dots,s; \text{ for } k+1,\dots,l) \tag{16}$$

and

$$G^{lk} = \frac{\partial C^l}{\partial y^k} \qquad (\text{for } l=1,\dots,s; \text{ for } k=1,\dots,l) \tag{17}$$

Numerical Results

The optimization of an eastern hemisphere geostationary mission is presented as an example. This mission, illustrated in Fig. 4, has two basic flight regimes and is typical of the problems that can be optimized efficiently with this approach. A general problem statement and the two formulations are

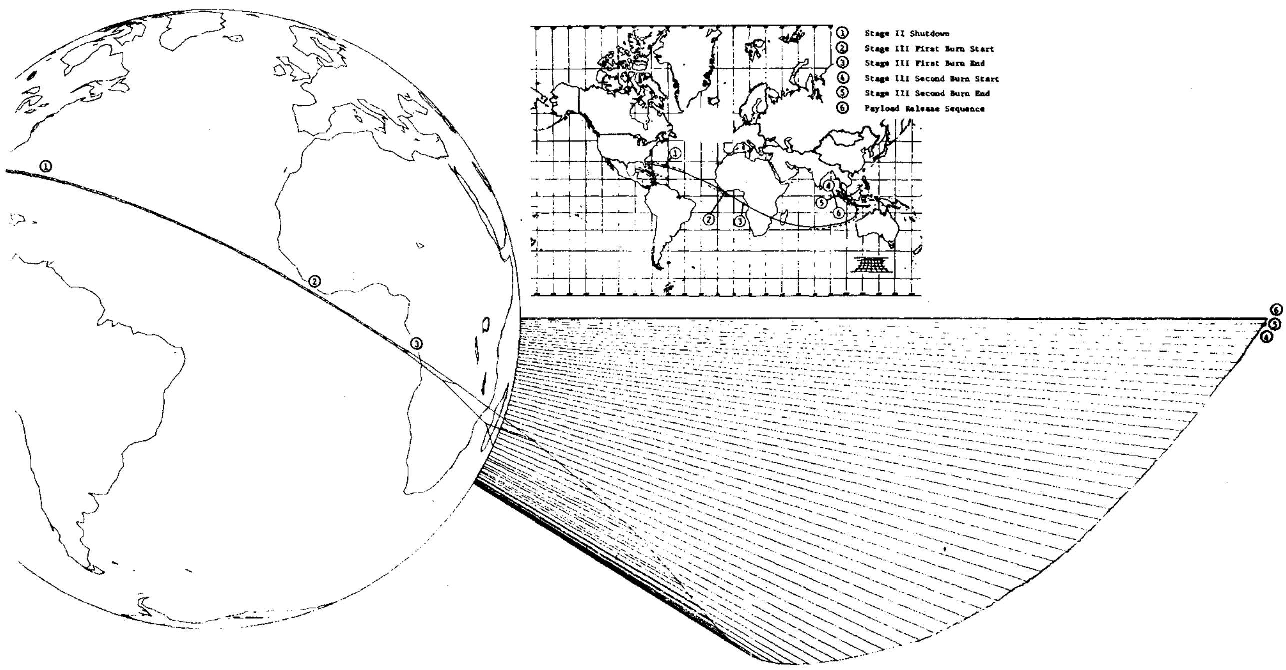

Fig. 4 Typical eastern hemisphere geostationary mission in Earth-centered relative coordinates.

given in Table 1. The standard nondecomposition formulation is a constrained discrete parameter optimization problem containing eight constraints and 10 independent variables. Let this problem be denoted as P1(8×10). As indicated, the complete mission was decomposed into three trajectory segments: ascent, apogee raising burn, and final circulation and plane change.

This problem could be decomposed differently; for example, the two orbital segments could be combined easily into a single subproblem. The master problem, as formulated, has only one constraint and three control variables; let the problem be denoted as MP1(1×3). It is important to note that, in MP1(1×3), two of the three control variables are constraint values in the subproblems. This enables the master algorithm to optimize the burnout velocity of the ascent leg and the inclination of the transfer orbit. These are two important geostationary mission design variables. The constraint that zero propellant remains at final burnout is really the active inequality constraint $W_{pr} \geq 0$. This is to insure that the program will not simulate consumption of more propellant than can be loaded into the tanks. This constraint is defined at the master level because it couples the subproblems in that stage III is used in all three of the trajectory segments. The subproblems, represented as SP1(3×3), SP2(2×2), and SP3(4×4), are all relatively small targeting problems.

Clearly, this sequence of subproblems is much easier to solve than P1(8×10). However, a computational tradeoff is involved because the sequence of subproblem $\{SP_i: i = 1,2,...S = 3\}$ must be solved on every trial step in univariate searches used to solve MP1(1×3). This fact emphasizes the importance of rapid subproblem solution and makes the addition of subproblem optimization questionable.

Computational results for each formulation are presented in Tables 2 and 3. The accelerated gradient projection option in the program to optimize simulated trajectories[8] was used to solve the standard formulation. On iterations 3 and 7, it had to be restarted and did not converge in ten iterations. In

Table 1 Standard and decomposition formulation of the geostationary mission

Determine the attitude steering history and burn times that maximize the payload a Titan IIIC class vehicle can deliver to an eastern hemisphere geostationary orbit. Assume that the elements for the optimum park orbit and transfer orbit are unknown and are to be determined.

Nondecomposition Formulation

Maximize: W_{pld} - Payload

Subject to the constraints:

$r^1 = 2.141197 \times 10^7$ ft	- Ascent burnout radius
$\gamma^1 = 0.0$ deg	- Ascent burnout flight path
$h_a^2 = 19{,}323$ n mi	- Transfer apogee altitude
$V^3 = 10{,}087$ fps	- SEO velocity
$\gamma^3 = 0.0$ deg	- SEO flight path angle
$\Lambda^3 = 90.0$ deg	- SEO flight azimuth
$\phi^3 = 0.0$ deg	- SEO latitude
$W_{pr}{}^3 = 0.0$ lb	- Propellant remaining

with the independent variables

W_{pld}	- Payload
$\dot{\theta}_{11}$	- Initial Stage I pitchover rate
$\dot{\theta}_{12}$	- 2nd Stage I pitch rate
$T_3{}^1$	- Duration of Stage III burn to achieve park orbit
ψ^2	- Yaw angle during transfer burn
$T_3{}^2$	- Duration of transfer burn
ϕ^3	- Latitude to initialize circularization burn
ψ^3	- Yaw angle during circularization burn
θ^3	- Pitch angle during circularization burn
$T_3{}^3$	- Duration of circularization burn

Decomposition Formulation

Master Problem:

Maximize: W_{pld} - Payload

Subject to: $W_{pr}{}^3 = 0.0$ lb - Propellant remaining

with the independent variables

W_{pld}	- Payload
$V_d{}^1$	- Ascent burnout velocity
$i_d{}^2$	- Inclination of the transfer orbit

Subproblem 1: Determine the values of $\dot{\theta}_{11}$, $\dot{\theta}_{12}$, and $T_3{}^1$ that solve the equations

$r^1 = 2.141197 \times 10^7$ ft	- Fixed
$V^1 = V_d{}^1$ fps	- Supplied by master algorithm
$\gamma^1 = 0.0$ deg	- Fixed

Subproblem 2: Determine the values of ψ^2, $T_3{}^2$ that solve the equations

$h_a{}^2 = 19{,}323$ n mi	- Fixed
$i^2 = i_d{}^2$ deg	- Supplied by master algorithm

Subproblem 3: Determine the values of ϕ^3, ψ^3, θ^3, and $T_3{}^3$ that solve the equations

$V^3 = 10{,}087$ fps	- Fixed
$\gamma^3 = 0.0$ deg	- Fixed
$\Lambda^3 = 90.0$ deg	- Fixed
$\phi^3 = 0.0$ deg	- Fixed

Table 2 Decomposition iteration summary

	Key Optimization Parameters			Key Trajectory Variable		
Master Level Iteration	Optimization Variable, Payload, lb	Constraint Error $\|\|C\|\|^2$	Optimization Indicator, deg	Park Orbit $h_p \times h_a$ n mi x n mi	Transfer Incl, deg	Subproblem Iterations
0	2500.00	1.15E-8	--	80.0 x 101.5	28.6	2, 3, 3
1	2674.80	2.89E-2	54.44	--	29.37	4, 6, 5
2	2773.91	4.33E-3	79.72	80.0 x 90.7	25.79	10, 17, 14
3	2781.03	1.08E-2	87.60	--	26.58	6, 10, 8
4	2782.67	1.04E-3	86.19	65.6 x 80.0	26.28	10, 9, 9
5	2783.70	3.85E-5	88.01	--	26.58	7, 9, 7
6	2784.71	2.09E-4	--	44.5 x 80.0	26.494	13, 10, 11

Table 3 Nondecomposition iteration summary

	Key Optimization Parameters			Key Trajectory Variable	
Iteration	Optimization Variable, Payload, lb	Constraint Error $\|\|e\|\|^2$	Optimization Indicator, deg	Park Orbit $h_p \times h_a$ n mi x n mi	Transfer Inclination, deg
0	2500.00	2.79E+5	--	--	--
1	2507.20	1.03E+5	--	--	--
2	2507.30	1.20E+3	--	--	--
3	2507.33	1.33E-3	2.12	--	--
4	2757.82	3.28E-3	--	--	--
5	2757.55	6.23E+1	--	-99 x 80.02	26.928
6	2751.48	4.63E-2	8.18	--	--
7	3030.51	1.09E+6	--	--	--
8	3029.80	1.89E+5	--	--	--
9	3029.43	1.49E+5	--	--	--
10	3028.56	1.19E+5	--	80.02 x 110	26.14

comparison, the decomposition algorithm did not require manual restarting and converged in only six iterations. The increased reliability is probably more significant than the reduction in the number of iterations. This is because each iteration at the master level represents considerably more computational effort than does each iteration in the standard formulation.

Summary

A new trajectory decomposition approach that does not require relaxation of state variable continuity between subproblems was presented. This approach enables complex multiple flight-regime missions to be optimized in a practical manner. Efficiency is obtained by partitioning the total mission design problem into more tractible subproblems. The subproblems were formulated as full-rank targeting problems. As a result, the complete subproblem solution process requires only the serial solution of a sequence of nonlinear equation sets. Each set of equations in this sequence is relatively small in dimension and can be solved quite rapidly. In contrast, other decomposition methods result in subproblems that are themselves smaller optimization problems. This is computationally significant because nonlinear equations can, in general, be solved more reliably and quickly than nonlinear programs of the same dimension.

Analytical equations for calculating sensitivities of master-level dependent variables to subproblem target values also are presented. These equations eliminated the need to run perturbed targeted trajectories to determine the master-level gradient vectors. These equations make this decomposition strategy computationally feasible.

Several extensions of this approach are possible. First, the addition of inequality constraints to the master-level problem would be useful. This could be accomplished easily merely by using a nonlinear programming version of the gradient-projection method at the master level. Second, relaxation of the full-rank subproblem restriction also would provide valuable user flexibility. For example, the minimum norm constraint restoration algorithm could be used to solve the subproblems. However, it is not clear how analytical partial derivatives at the master level could be obtained is this were allowed. Further work is required in this area. Finally, the use of guidance steering algorithms to solve the subproblems would be very efficient. This would eliminate the need for the iterative solution of the subproblems via the Newton-Raphson algorithm. For example, simple explicit guidance algorithms, such as linear sine or linear tangent, could be used to satisfy the subproblem constraints. These techniques, though perhaps suboptional, are quite efficient and, in a sense, represent a better approximation to the actual vehicle steering capabilities.

Acknowledgment

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